

ON SUPERCRITICAL COUPLED ELLIPTIC SYSTEMS WITH POWER NONLINEARITIES

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ABSTRACT. Consider the following coupled elliptic system of equations

$$(-\Delta)^s u_i = |u|^{p-1} u_i \quad \text{in } \mathbb{R}^n$$

where $0 < s \leq 2$, $p > 1$, $m \geq 1$, $u = (u_i)_{i=1}^m$ and $u_i : \mathbb{R}^n \rightarrow \mathbb{R}$. The qualitative behaviour of solutions of the above system has been studied from various perspectives in the literature including the free boundary problems and the classification of solutions for Sobolev sub- and super-critical exponents. In this paper, we derive monotonicity formulae for entire solutions of the above local, when $s = 1, 2$, and nonlocal, when $0 < s < 1$ and $1 < s < 2$, system. Then we apply these to give a classification of finite Morse index solutions. Note that there is no sign assumption on solutions. For the case of scalar equations, that is when $m = 1$, such a classification is given by Farina in [17], Davila, Dupaigne, Wang and Wei in [12], Davila, Dupaigne and Wei in [11] and Wei and the first author in [21] for $s = 1$, $s = 2$, $0 < s < 1$ and $1 < s < 2$, respectively. In our knowledge, this is the first work regarding classifications of finite Morse index solutions of the above system of equations.

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1. INTRODUCTION

1.1. Background. Our objective in this paper is to classify finite Morse index solutions of the following coupled elliptic system of equations

$$(1.1) \quad (-\Delta)^s u_i = |u|^{p-1} u_i \quad \text{in } \mathbb{R}^n$$

where $0 < s \leq 2$, $p > 1$, $m \geq 1$, $u = (u_i)_{i=1}^m$ and $u_i : \mathbb{R}^n \rightarrow \mathbb{R}$.

An important feature of system (1.1) is the variational structure of the coupled equations, i.e.

$$(1.2) \quad -\Delta u = \nabla H(u) \quad \text{in } \mathbb{R}^n,$$

where in our case $H(u) = \frac{1}{p+1} |u|^{p+1}$. Due to this variational structure, qualitative behaviour of solutions of system (1.2) has been studied extensively in the context of elliptic partial differential equations from both pure and applied mathematics perspectives. Let us mention that Andersson, Shahgholian, Uraltseva, and Weiss in [2] considered system (1.1), up to a negative sign, when $p = 0$ and $s = 1$ which reads

$$\Delta u_i = \frac{u_i}{|u|} \chi_{\{|u|>0\}}.$$

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This is closely related to minimizers of the energy

$$\int \left[\sum_{i=1}^m |\nabla u_i|^2 + 2|u| \right] dx,$$

and they studied the regularity of free boundaries. To prove regularity results in [2], they established a monotonicity formula that is inspired by the one given by Weiss in [41, 42]. We refer interested readers to the book of Petrosyan, Shahgholian, Uraltseva in [33] for more information.

It should be remarked that as a particular case, one can consider the case of two particles, that is $m = 2$, and $p = 3$ namely

$$(1.3) \quad \begin{cases} (-\Delta)^s u &= (u^2 + v^2)u \text{ in } \mathbb{R}^n, \\ (-\Delta)^s v &= (u^2 + v^2)v \text{ in } \mathbb{R}^n. \end{cases}$$

Note that this system is a special case of the nonlinear Schrödinger system that is

$$(1.4) \quad \begin{cases} (-\Delta)^s u &= (\mu_1 u^2 + \beta v^2)u \text{ in } \mathbb{R}^n, \\ (-\Delta)^s v &= (\mu_2 u^2 + \beta v^2)v \text{ in } \mathbb{R}^n, \end{cases}$$

where μ_1, μ_2, β are constants. We would like to mention that most of the results in the paper are valid for (1.4) as well. We refer interested readers to [31, 38, 39] and references therein for more information regarding classification of solutions of system (1.4) for various parameters μ_1, μ_2 and β .

1.2. Tools and methodology (Historical development). Semilinear elliptic equations (with almost four decades of history) are probably the most widely and intensely studied equations in PDE. The simplest equation in this class is expressed as

$$\Delta u = f(u),$$

and exhibits as many possible features, as the right hand side nonlinearity $f(u)$ may do. This equation has also been studied from so many different perspectives, that any attempt to try to list them here would probably fail. There are however two different types of questions, of interest to the current authors, that seem to be interconnected and developed in parallel, but almost separately:

- i) Analysis of the singular set: $\{x : \nabla u = 0\}$.
- ii) Stability and Liouville's type questions.

One can actually add a third less studied, and probably more complicated problem to the above, which is:

- iii) Structure of the blow-up set: $\{x : |u(x)| = \infty\}$.

In studying the above problems, experts have (in many cases) developed parallel tools and ideas to handle technical passages of the analysis of questions in each cases. Two of these tools, also utilized in this paper, are *Monotonicity functional*, and *Blow-up/down Analysis*.

The particular monotonicity functional used here combines the energy functional along with a balanced term (see below for explicit form). The use of monotonicity functional (in the way we present here) can be traced back¹ to the work of H.W.

¹It is worth mentioning that a different type of almost monotonicity functional was used by Arne Beurling in his thesis: A. Beurling, Etudes sur un probleme de majoration, thesis,

Fleming in [22] for area minimizing currents which has subsequently been proved by Allard in [1] for stationary rectifiable varifolds, that was later developed further by others. The monotonicity functional that we shall use in this paper originates in the study of harmonic maps by Price in [34] and Schoen and Uhlenbeck in [37], see also [36]. And in connection to free boundary problems, and it is first used by B. Ou [3], and developed later by G.S. Weiss in [42] and others in various forms. In the context of semilinear heat equations, a similar monotonicity formula is given by Giga and Kohn in [25] and for the corresponding elliptic equation by Pacard [32].

The second main tool, blow-up/down analysis, has its origin in local regularity theory and the so-called linearization technique (also called harmonic blow-up). To study local structure of level surfaces of a solution to the above semilinear problem, one considers scaled version of the problem and classifies the limit manifold of such scalings. The limit manifold naturally carries information of the local behavior of the solutions and hence one can then with some devices link back this behavior to the local problem and deduce the expected properties. It is now that the role of monotonicity functional become crucial and indispensable in that one can use the fact that when blowing up a solution, the monotonicity functional, call it $E(r, u)$, being monotone will have a limit. At the same time this functional has a nice scaling properties $E(rs, u) = E(s, u_r)$ for $r, s > 0$ and u_r a correctly scaled version of u . Hence one obtains

$$E(0^+, u) = \lim_{r \rightarrow 0} E(rs, u) = \lim_{r \rightarrow 0} E(s, u_r) = E(s, u_0),$$

so that $E(s, u_0)$ is constant. Now a strong version of the monotonicity functional asserts that the only time E is constant is when the function u_0 is homogeneous, where the order of homogeneity is dictated by E . Since the same can be done by blowing-down, we will have that u_0, u_∞ are homogeneous of same order. Next if one can prove that the homogenous solutions are unique we must then have $u_0 = u_\infty$ and hence $E(0^+, u) = E(\infty, u)$, and we arrive at $E(r, u) = \text{constant}$. Therefore u is itself homogeneous, and already classified.

In the regularity theory of level surfaces the classification of homogeneous global solutions is one of the key elements of the theory, and many time a very hard nut to crack, in higher dimensions; in two dimensions homogenous solutions can be computed directly. In the stability theory and Liouville type problems this part is based on computations of eigenvalues of the corresponding Laplace-Beltrami on the sphere, as also done in this paper. As shown here below (and in many preceding papers by experts in this area) the only homogenous solutions in the *appropriate* space are the trivial solutions. In particular the energy functional becomes zero and hence $E(r, u) = 0$. From here one deduces that $u \equiv 0$ is the only solution satisfying the given condition.

It is worthwhile remarking the fundamental feature of these problems, that in some cases depend on the dimension and order of homogeneity are directly a consequence of the eigenvalues of Laplace-Beltrami, where in the case of index-theory results in the computation of eigenvalues for nonlinear Laplace-Beltrami, that in turn implies that non-trivial solutions do not exists in certain range of values. These

(Uppsala), 1933. The monotonicity functional of A. Beurling states that for a jordan curve γ , the product of the harmonic measures for both sides of the curve can be controlled as follows: $\omega_1(B_r(z) \cap \gamma) \cdot \omega_2(B_r(z) \cap \gamma) \leq Ar^2$, where $z \in \gamma$, and ω_1, ω_2 denote the harmonic measures on each side of the curve.

ideas have their origin in the regularity theory of minimal surfaces, that boils down to proving minimal cones do not exist in dimensions less than eight.

1.3. Problem setting. Our main technique is to derive monotonicity formulae for solutions of (1.1) for various values of parameter $0 < s \leq 2$. To provide such monotonicity formulae we consider various cases.

Case $s = 1$. Consider the following functional for every $\lambda > 0$ and $x_0 \in \mathbb{R}^n$

$$(1.5) \quad E_1(u, \lambda, x_0) := \lambda^{-n+2\frac{p+1}{p-1}} \int_{B_\lambda(x_0)} \frac{1}{2} \sum_{i=1}^m |\nabla u_i|^2 - \frac{1}{p+1} |u|^{p+1}$$

$$(1.6) \quad + \frac{1}{p-1} \lambda^{-n+2\frac{p+1}{p-1}-1} \int_{\partial B_\lambda(x_0)} |u|^2,$$

then the following monotonicity formula holds for classical solutions of (1.1).

Theorem 1.1. *Suppose that u is a solution of (1.1) for $s = 1$. Then, E_1 is a nondecreasing function of λ and in fact*

$$(1.7) \quad \frac{dE_1}{d\lambda} = \lambda^{-n+1+2\frac{p+1}{p-1}} \int_{\partial B_\lambda(x_0)} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial r} + \frac{2}{p-1} \frac{u_i}{r} \right)^2,$$

where $\frac{\partial}{\partial r}$ is polar derivative.

Note that for the case of single equations, that is when $m = 1$, similar monotonicity formulae are given by Pacard in [32] and Weiss in [41, 42]. For the case of systems that is when $m \geq 1$, very recently, Andersson, Shahgholian, Uraltseva and Weiss in [2] provided a monotonicity formula for solutions of system (1.1) when $p = 0$ and applied it to study free boundary problems. Let us mention that very similar monotonicity formulae appear in the field of harmonic maps that is $-\Delta u = |\nabla u|^2 u$ where $u : \mathbb{R}^n \rightarrow \mathbb{S}^{m-1}$, see Evans in [15, 16] and reference therein.

Case $s = 2$. For every $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ define

$$(1.8) \quad E_2(u, \lambda, x_0) := \lambda^{4\frac{p+1}{p-1}-n} \int_{B_\lambda(x_0)} \left(\frac{1}{2} \sum_{i=1}^m |\Delta u_i|^2 - \frac{1}{p+1} |u|^{p+1} \right) \\ - \frac{4}{p-1} \left(\frac{p+3}{p-1} - n \right) \lambda^{1+\frac{s}{p-1}-n} \int_{\partial B_\lambda(x_0)} |u|^2 \\ - \frac{4}{p-1} \left(\frac{p+3}{p-1} - n \right) \frac{d}{d\lambda} \left[\lambda^{\frac{s}{p-1}+2-n} \int_{\partial B_\lambda(x_0)} |u|^2 \right] \\ + \frac{1}{2} \lambda^3 \frac{d}{d\lambda} \left[\lambda^{\frac{s}{p-1}+1-n} \int_{\partial B_\lambda(x_0)} \sum_{i=1}^m \left(\frac{4}{p-1} \lambda^{-1} u_i + \frac{\partial u_i}{\partial r} \right)^2 \right] \\ + \frac{1}{2} \frac{d}{d\lambda} \left[\lambda^{4\frac{p+1}{p-1}-n} \int_{\partial B_\lambda(x_0)} \sum_{i=1}^m \left(|\nabla u_i|^2 - \left| \frac{\partial u_i}{\partial r} \right|^2 \right) \right] \\ + \frac{1}{2} \lambda^{4\frac{p+1}{p-1}-n-1} \int_{\partial B_\lambda(x_0)} \sum_{i=1}^m \left(|\nabla u_i|^2 - \left| \frac{\partial u_i}{\partial r} \right|^2 \right).$$

Then the following monotonicity formula holds for the fourth order Lane-Emden system.

Theorem 1.2. *Suppose that $n \geq 5$, $p > \frac{n+4}{n-4}$ and u is a solution of (1.1) when $s = 2$. For any $\lambda > 0$ and $x_0 \in \mathbb{R}^n$*

$$(1.9) \quad \frac{dE_2(u, \lambda, x_0)}{d\lambda} \geq C\lambda^{\frac{s}{p-1}+2-n} \int_{\partial B_\lambda(x_0)} \sum_{i=1}^m \left(\frac{4}{p-1} r^{-1} u_i + \frac{\partial u_i}{\partial r} \right)^2$$

where E_2 is defined by (1.8) and C is independent from λ .

Note also that the above monotonicity formula is provided by Davila, Dupaigne, Wang and Wei in [12] for the case of single equations that is when $m = 1$. For the case of fractional Laplacian, we provide monotonicity formulae for the extension problems. Assume that $u_i \in C^{2\sigma}(\mathbb{R}^n)$, $\sigma > s > 0$ and

$$\int_{\mathbb{R}^n} \frac{|u_i(y)|}{(1+|y|)^{n+2s}} dy < \infty,$$

for each $1 \leq i \leq m$. The fractional Laplacian of u_i when $0 < s < 1$ denoted by

$$(1.10) \quad (-\Delta)^s u_i(x) := p.v. \int_{\mathbb{R}^n} \frac{u_i(x) - u_i(y)}{|x - y|^{n+2s}} dy,$$

is well-defined for every $x \in \mathbb{R}^n$. Here *p.v.* stands for the principle value. It is by now standard that the fractional Laplacian can be seen as a Dirichlet-to-Neumann operator for a degenerate but local diffusion operator in the higher-dimensional half-space \mathbb{R}^{n+1} , see Caffarelli and Silvestre in [5]. In other words, for $u_i \in C^{2\sigma} \cap L^1(\mathbb{R}^n, (1+|y|^{n+2s})dy)$ when $\sigma > s$ and $0 < s < 1$, there exists $v = (v_i)_{i=1}^m$ such that $v_i \in C^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$, $y^{1-2s} \partial_y v_i \in C(\overline{\mathbb{R}_+^{n+1}})$ and

$$(1.11) \quad \begin{cases} \nabla \cdot (y^{1-2s} \nabla v_i) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ v_i = u_i & \text{on } \partial \mathbb{R}_+^{n+1}, \\ -\lim_{y \rightarrow 0} y^{1-2s} \partial_y v_i = \kappa_s |v|^{p-1} v_i & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

where

$$(1.12) \quad \kappa_s = \frac{\Gamma(1-s)}{2^{2s-1} \Gamma(s)}.$$

For the case of $1 < s < 2$ there are various definitions for the fractional operator $(-\Delta)^s$, see [6, 9, 21, 44]. From the Fourier transform one can define the fractional Laplacian by

$$\widehat{(-\Delta)^s u_i}(\zeta) = |\zeta|^{2s} \hat{u}_i(\zeta)$$

or inductively by $(-\Delta)^s = (-\Delta)^{s-1} o(-\Delta)$. Note that Yang in [44] gave a counterpart of the extension problem by Caffarelli and Silvestre in [5] for the fractional Laplacian $(-\Delta)^s$, where s is any positive, noninteger number. In other words, he showed that the higher order fractional Laplacian operator can also be regarded as the Dirichlet-to-Neumann map for an extension function satisfying a higher order elliptic equation in the upper half space with one extra spatial dimension. More precisely, there exists an extension function $v_i \in W^{2,2}(\mathbb{R}_+^{n+1}, y^b)$ such that

$$(1.13) \quad \begin{cases} \Delta_b^2 v_i = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} y^b \partial_y v_i = 0 & \text{in } \partial \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} y^b \partial_y \Delta_b v_i = C_{n,s} |v|^{p-1} v_i & \text{in } \mathbb{R}^n, \end{cases}$$

where $b := 3 - 2s$ and $\Delta_b v_i := y^{-b} \operatorname{div}(y^b \nabla v_i)$. We refer interested readers to Case and Chang [6] and Chang and Gonzales [9] as well. We are now ready to provide the monotonicity formulae.

Case $0 < s < 1$. Let $v = (v_i)_{i=1}^m$ be a solution of the extension problem (1.11). Now define the energy functional for any $\lambda > 0$ and $x_0 \in \partial \mathbb{R}_+^{n+1}$

$$(1.14) \quad E_s(v, \lambda, x_0) \\ := \lambda^{\frac{2s(p+1)}{p-1}-n} \left(\frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B_\lambda} y^{1-2s} \sum_{i=1}^m |\nabla v_i|^2 dx dy - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_\lambda} |v|^{p+1} dx \right) \\ + \lambda^{\frac{2s(p+1)}{p-1}-n-1} \frac{s}{p+1} \int_{\partial B_\lambda \cap \mathbb{R}_+^{n+1}} y^{1-2s} |v|^2 d\sigma.$$

Theorem 1.3. Suppose that $0 < s < 1$. Let $v = (v_i)_{i=1}^m$ where each $v_i \in C^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$ be a solution of (1.11) such that $y^{1-2s} \partial_y v_i \in C(\overline{\mathbb{R}_+^{n+1}})$. Then, E_s is a nondecreasing function of λ . Furthermore,

$$(1.15) \quad \frac{dE_s}{d\lambda} = \lambda^{\frac{2s(p+1)}{p-1}-n+1} \int_{\partial B(x_0, \lambda) \cap \mathbb{R}_+^{n+1}} y^{1-2s} \sum_{i=1}^m \left(\frac{\partial v_i}{\partial r} + \frac{2s}{p-1} \frac{v_i}{r} \right)^2 d\sigma,$$

where E_s provided in (1.14).

Case $1 < s < 2$. Suppose that $v = (v_i)_{i=1}^m$ is a solution of the extension problem (1.13). Similarly, now define the energy functional for any $\lambda > 0$ and $x_0 \in \partial \mathbb{R}_+^{n+1}$

$$(1.16) \quad E_s(v, \lambda, x_0) \\ := \lambda^{2s \frac{p+1}{p-1}-n} \left(\int_{\mathbb{R}_+^{n+1} \cap B_\lambda(x_0)} \frac{1}{2} y^{3-2s} \sum_{i=1}^m |\Delta_b v_i|^2 - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_\lambda(x_0)} |v|^{p+1} \right) \\ - \frac{s}{p-1} \left(\frac{p+2s-1}{p-1} - n \right) \lambda^{-3+2s+\frac{4s}{p-1}-n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^{3-2s} |v|^2 \\ - \frac{s}{p-1} \left(\frac{p+2s-1}{p-1} - n \right) \frac{d}{d\lambda} \left[\lambda^{\frac{4s}{p-1}+2s-2-n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^{3-2s} |v|^2 \right] \\ + \frac{1}{2} r^3 \frac{d}{d\lambda} \left[\lambda^{\frac{4s}{p-1}+2s-3-n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^{3-2s} \sum_{i=1}^m \left(\frac{2s}{p-1} \lambda^{-1} v_i + \frac{\partial v_i}{\partial r} \right)^2 \right] \\ + \frac{1}{2} \frac{d}{d\lambda} \left[\lambda^{2s \frac{p+1}{p-1}-n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^{3-2s} \sum_{i=1}^m \left(|\nabla v_i|^2 - \left| \frac{\partial v_i}{\partial r} \right|^2 \right) \right] \\ + \frac{1}{2} \lambda^{2s \frac{p+1}{p-1}-n-1} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^{3-2s} \sum_{i=1}^m \left(|\nabla v_i|^2 - \left| \frac{\partial v_i}{\partial r} \right|^2 \right).$$

Theorem 1.4. Assume that $n > 2s$ and $p > \frac{n+2s}{n-2s}$. Let $v = (v_i)_{i=1}^m$ be a solution of (1.13) then $E(v, \lambda, x_0)$ is a nondecreasing function of $\lambda > 0$. In addition,

$$(1.17) \quad \frac{dE_s(v, \lambda, x_0)}{d\lambda} \geq C(n, s, p) \lambda^{\frac{4s}{p-1}+2s-2-n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_\lambda(x_0)} y^{3-2s} \sum_{i=1}^m \left(\frac{2s}{p-1} r^{-1} v_i + \frac{\partial v_i}{\partial r} \right)^2,$$

where E_s is given by (1.16) and $C(n, s, p)$ is independent from λ .

We also refer interested readers to Davila, Dupaigne and Wei in [11] and to Wei and the first author in [21] for a similar monotonicity formula for the case of scalar equations and $0 < s < 1$ and $1 < s < 2$, respectively. Before we state our main results let us present some backgrounds regarding classification of solutions of (1.1) in the absence of stability. We provide such classifications for scalar equations and systems separately.

Remark 1.1. *Note that monotonicity formulae provided as Theorems 1.1-1.4 hold for the following system with a slightly more general right-hand side,*

$$(1.18) \quad (-\Delta)^s u_i = |u|^{p-1} (\alpha_i u_i^+ + \beta_i u_i^-) \quad \text{in } \mathbb{R}^n,$$

where α_i and β_i are positive constants for $1 \leq i \leq m$.

Suppose that $m = 1$ when (1.1) turns into a single equation. We first consider the local operator cases meaning $s = 1$ and $s = 2$. Suppose that $s = 1$ and the parameter p is in the subcritical case that is when $1 < p < p_S(n, 1)$ where

$$p_S(n, 1) := \begin{cases} \infty & \text{if } n \leq 2, \\ \frac{n+2}{n-2} & \text{if } n > 2. \end{cases}$$

For this case, there is a very well-known classification result of Gidas and Spruck in [23] stating that the only nonnegative solution of the Lane-Emden equation (1.1) with $s = 1$ is the trivial solution. For the critical case, that is when $p = p_S(n, 1)$, Caffarelli, Gidas and Spruck in [4] that there is a unique (up to translation and rescaling) positive solution for the Lane-Emden equation. For the fourth order Lane-Emden equation, that is when $s = 2$, Wei and Xu in [40] provided a similar classification result for the subcritical $1 < p < p_S(n, 2)$ and the critical cases $p = p_S(n, 2)$ when

$$p_S(n, 2) := \begin{cases} \infty & \text{if } n \leq 4, \\ \frac{n+4}{n-4} & \text{if } n > 4, \end{cases}$$

see also [30]. Note that for the fractional Laplacian operator $0 < s < 1$, such a classification result is given by Li [29] and Chen-Li-Ou [8] where the critical exponent is

$$(1.19) \quad p_S(n, s) = \begin{cases} \infty & \text{if } n \leq 2s, \\ \frac{n+2s}{n-2s} & \text{if } n > 2s. \end{cases}$$

For the case of system of equations, that is when $m \geq 2$, Druet, Hebey and Vétois in [14] provided a classification result for solutions of (1.1) where $s = 1$ via the moving sphere method. Suppose that $p = \frac{n+2}{n-2}$ and $u = (u_i)_{i=1}^m$ is a nonnegative classical solution of (1.1) where $s = 1$. Then they proved that there exist $x_0 \in \mathbb{R}^n, \lambda > 0, \Lambda \in \mathcal{S}_+^{m-1}$ such that

$$(1.20) \quad u(x) = \left(\frac{\lambda}{\lambda^2 + \frac{1}{n(n-2)}|x - x_0|^2} \right)^{\frac{n-2}{2}} \Lambda.$$

We also refer interested readers to [13, 26, 27] where the authors studied various counterparts of system (1.1).

In this paper, we are interested in classifying finite Morse index solutions of (1.1). Before we state our main results, let us fix the following notation and provide a definition thereafter.

Notation 1.1. The seminorm $||\cdot||_{H^s(\mathbb{R}^n)}$ is defined as what follows for various cases, when $h : \mathbb{R}^n \rightarrow \mathbb{R}$.

- *Local Cases:* For $s = 1$

$$(1.21) \quad ||h||_{\dot{H}^1(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |\nabla h|^2 dx \right]^{1/2},$$

and for the case of $s = 2$

$$(1.22) \quad ||h||_{\dot{H}^2(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |\Delta h|^2 dx \right]^{1/2}.$$

- *Nonlocal Case:* When $0 < s < 1$ and $1 < s < 2$ define the seminorm

$$(1.23) \quad ||h||_{\dot{H}^s(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(h(x) - h(y))^2}{|x - y|^{n+2s}} dx dy \right]^{1/2}.$$

Definition 1.1. We say a solution u of (1.1) is stable outside a compact set if there exists $R_0 > 0$ such that

$$(1.24) \quad \sum_{i=1}^m \int |u|^{p-1} \phi_i^2 + (p-1) \sum_{i,j=1}^m \int |u|^{p-3} u_i u_j \phi_i \phi_j \leq \sum_{i=1}^m ||\phi_i||_{\dot{H}^s(\mathbb{R}^n)}^2,$$

for any $\phi = (\phi_i)_{i=1}^m$ where $\phi_i \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_0}})$ for $1 \leq i \leq m$.

Here is our main result.

Theorem 1.5. Suppose $0 < s \leq 2$ and $n > 2s$. Let u be a solution of (1.1) that is stable outside a compact set. Then either for $1 < p < p_S(n, s)$ or for $p > p_S(n, s)$ and

$$(1.25) \quad p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

each component u_i must be identically zero. For the case of Sobolev critical exponent, that is when $p = p_S(n, s)$, a solution u has finite energy that is

$$||u||_{L^{p+1}(\mathbb{R}^n)}^{p+1} = ||u||_{\dot{H}^s(\mathbb{R}^n)}^2 < \infty.$$

In this case, if in addition u is stable, then each component u_i must be identically zero.

To clarify the result, the above theorem implies that the only nonnegative solution for system (1.3) when $s = 1$ is the trivial solution for $n < 12$ and $n \neq 4$.

We finish this section with the fact that the following singular function

$$(1.26) \quad u_s(x) = \mathcal{A}|x|^{-\frac{2s}{p-1}},$$

where $\mathcal{A} \in \mathbb{R}^m$ for

$$(1.27) \quad |\mathcal{A}|^{p-1} = 2^{2s} \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})},$$

solves (1.1). Here we assumed that the parameter p is supercritical that is $p > p_S(n, s)$. In upcoming sections we provide a proof for Theorem 1.5.

2. LOCAL CASE: LAPLACIAN OPERATOR

In this section we assume that $s = 1$. Therefore system (1.1) turns into the following form

$$(2.1) \quad -\Delta u_i = |u|^{p-1} u_i \quad \text{in } \mathbb{R}^n.$$

Note that this is a particular case of system (1.2). For a general nonlinearity $H : \mathbb{R}^m \rightarrow \mathbb{R}$ such that $\nabla H \geq 0$, it is proved in [19] that bounded stable solutions of (1.2) are constant when $n \leq 4$. For radial solutions, this is known to hold in more dimensions that is when $n \leq 9$ without any sign assumptions on the nonlinearity, see [18]. In addition it is known that at least for the case of radial solutions the dimension $n = 9$ is the optimal dimension. For the rest of this section we prove the following classification of finite Morse index solutions of (2.1).

Theorem 2.1. *Suppose that $u = (u_i)_{i=1}^m$ is a finite Morse index solution of (2.1) when $m \geq 1$ and $n \geq 3$. Let $1 < p < \frac{n+2}{n-2}$ and $\frac{n+2}{n-2} < p < p_c(n)$ where*

$$(2.2) \quad p_c(n) = \begin{cases} \infty & \text{if } n \leq 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11. \end{cases}$$

Then each u_i must be identically zero. For the Sobolev critical exponent $p = \frac{n+2}{n-2}$, a solution u has finite energy that is

$$\int_{\mathbb{R}^n} |u|^{p+1} = \sum_{i=1}^m \int_{\mathbb{R}^n} |\nabla u_i|^2 < \infty.$$

In this case, if in addition u is stable, then each component u_i must be identically zero.

Note that for the case of scalar equations, that is when $m = 1$, the above theorem is given by Farina in [17]. He used a Moser iteration type argument for the proof. We refer interested reader to Crandall and Rabinowitz in [10] for a similar approach. To prove the above theorem, we apply a blow-down analysis argument as well as the monotonicity formula given as Theorem 1.1. We now derive a few elliptic estimates.

Lemma 2.1. *Suppose that u is a stable solution of (1.1). Then for any $R > 1$ we have*

$$(2.3) \quad \int_{B_R} |u|^{p+1} \leq CR^{n-2\frac{p+1}{p-1}},$$

$$(2.4) \quad \sum_{i=1}^m \int_{B_R} |\nabla u_i|^2 \leq CR^{n-2\frac{p+1}{p-1}},$$

where C is a positive constant that is independent from R .

Proof. Test the stability inequality on $\phi_i = u_i \zeta_R$ where $\zeta_R : C_c^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ and $\zeta_R \equiv 1$ on B_R and $\zeta_R \equiv 0$ on $\mathbb{R}^n \setminus B_{2R}$ with $\|\zeta_R\|_{L^\infty(B_{2R} \setminus B_R)} \leq CR^{-1}$. Now multiply both sides of (1.1) with $u_i \zeta_R^2$ and integrate by parts. Equating the inequalities that one gets from this and from stability finishes the proof. \square

Applying the Hölder's inequality we get the following L^2 estimate.

Corollary 2.1. *Suppose that u is a stable solution of (1.1). Then for any $R > 1$ we have*

$$(2.5) \quad \int_{B_R} |u|^2 \leq CR^{n-\frac{4}{p-1}},$$

where C is a positive constant that is independent from R .

In this part, we present a classification of stable homogeneous solutions. This is a key point in our proof of Theorem 2.1.

Theorem 2.2. *Suppose that $u = (u_i)_{i=1}^m$ for $u_i = r^{-\frac{2}{p-1}} \psi_i(\theta)$ is a stable solution of (1.1). Then each ψ_i is identically zero provided $\frac{n+2}{n-2} < p < p_c(n)$ where $p_c(n)$ is given by (2.2).*

Proof. We omit the proof here, since a similar argument will be provided in the proof of Theorem 3.2 for the fourth order case. \square

Proof of Theorem 2.1. The proof is based on a blow-down analysis and it relies on the monotonicity formula provided as Theorem 1.1. We omit the details, since similar arguments will be provided for the poof of Theorem 3.1. For the case of Sobolev critical exponent, one can conclude the result via applying the Pohozaev identity. \square

3. LOCAL CASE: BI-LAPLACIAN OPERATOR

In this section, we consider the following fourth order Lane-Emden system

$$(3.1) \quad \Delta^2 u_i = |u|^{p-1} u_i \quad \text{in } \mathbb{R}^n.$$

This section is devoted to the proof of the following theorem.

Theorem 3.1. *Suppose that $u = (u_i)_{i=1}^m$ is a finite Morse index solution of (3.1) when $m \geq 1$ and $n \geq 5$. Let $1 < p < \frac{n+4}{n-4}$ and $\frac{n+4}{n-4} < p < \bar{p}_c(n)$ where*

$$(3.2) \quad \bar{p}_c(n) = \begin{cases} \infty & \text{if } n \leq 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geq 13. \end{cases}$$

Then each u_i must be identically zero. For the Sobolev critical exponent $p = \frac{n+4}{n-4}$, a solution u has finite energy that is

$$\int_{\mathbb{R}^n} |u|^{p+1} = \sum_{i=1}^m \int_{\mathbb{R}^n} |\Delta u_i|^2 < \infty.$$

In this case, if in addition u is stable, then each component u_i must be identically zero.

In order to prove the above theorem, we are required to establish some a priori estimates on solutions.

Lemma 3.1. *Suppose that $u = (u_i)_{i=1}^m$ be a smooth stable solution of (3.1) and set $w = (w_i)_{i=1}^m$ where $w_i = \Delta u_i$. Then the following estimate holds,*

$$(3.3) \quad \int_{\mathbb{R}^n} (|w|^2 + |u|^{p+1}) \zeta^2 \leq C \int_{\mathbb{R}^n} |u|^2 [|\nabla \Delta \zeta| |\nabla \zeta| + |\Delta \zeta|^2 + |\Delta(|\nabla \zeta|^2)|]$$

$$(3.4) \quad + C \int_{\mathbb{R}^n} |u| |w| |\nabla \zeta|^2,$$

for a test function $\zeta : C_c^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$.

Proof. Test the stability inequality on $u_i \zeta$ where ζ is a test function then multiply both sides of the i^{th} equation of (1.1) with $u_i \zeta^2$. Equating these completes the proof. \square

Applying an appropriate test function yields the following estimate.

Corollary 3.1. *Under the same assumptions as (3.1), there exists a constant C such that*

$$(3.5) \quad \int_{B_R(x)} |w|^2 + |u|^{p+1} \leq CR^{-4} \int_{B_{2R}(x) \setminus B_R(x)} |u|^2 + CR^{-2} \int_{B_{2R}(x) \setminus B_R(x)} |u| |w|,$$

and therefore

$$(3.6) \quad \int_{B_R(x)} |w|^2 + |u|^{p+1} \leq CR^{n-4\frac{p+1}{p-1}}.$$

We now provide classification of stable homogeneous solutions. Note first that the following Hardy-Rellich inequality with the best constant holds. Suppose that $h : C_c^2(\mathbb{R}^n) \rightarrow \mathbb{R}$ then

$$(3.7) \quad \int_{\mathbb{R}^n} |\Delta h|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{h^2}{|x|^4} dx.$$

This inequality implies that the singular solution given by (1.26) is stable if and only if

$$p|\mathcal{A}|^{p-1} = p\alpha(\alpha+2)(n-\alpha-2)(n-\alpha-4) \leq \frac{n^2(n-4)^2}{16},$$

where $\alpha := \frac{4}{p-1}$ and \mathcal{A} given by (1.27).

Theorem 3.2. *Suppose that $u = (u_i)_{i=1}^m$ for $u_i = r^{-\frac{4}{p-1}} \psi_i(\theta)$ is a stable solution of (3.1). Then each $\psi_i \equiv 0$ provided $\frac{n+4}{n-4} < p < \bar{p}_c(n)$ where $\bar{p}_c(n)$ given by (3.2).*

Proof. It is straightforward to see that $\psi = (\psi_i)_{i=1}^m$ satisfies

$$(3.8) \quad \Delta_\theta^2 \psi_i - \alpha \Delta_\theta \psi_i + \beta \psi_i = |\psi|^{p-1} \psi_i,$$

where

$$(3.9) \quad \alpha := (q+2)(n-4-q) + q(n-2-q),$$

$$(3.10) \quad \beta := q(q+2)(n-4-q)(n-2-q),$$

where $q := \frac{4}{p-1}$. Multiplying both sides with ψ_i and integrating over \mathcal{S}^{n-1} we get

$$(3.11) \quad \sum_{i=1}^m \int_{\mathcal{S}^{n-1}} [|\Delta \psi_i|^2 + \alpha |\nabla \psi_i|^2] + \beta \int_{\mathcal{S}^{n-1}} |\psi|^2 = \int_{\mathcal{S}^{n-1}} |\psi|^{p+1}.$$

Now applying the stability inequality (1.24) for a test function $\phi_i := r^{-\frac{n-4}{2}}\psi_i(\theta)\eta_\epsilon(r)$ for some appropriate cut-off function η_ϵ , one can get

$$(3.12) \quad p \int_{S^{n-1}} |\psi|^{p+1} \leq \sum_{i=1}^m \int_{S^{n-1}} \left[|\Delta \psi_i|^2 + \frac{n(n-4)}{2} |\nabla \psi_i|^2 \right] + \frac{n^2(n-4)^2}{16} \int_{S^{n-1}} |\psi|^2.$$

Combining (3.11) and (3.12) we get

$$(3.13) \quad (p-1) \sum_{i=1}^m \int_{S^{n-1}} \left[|\Delta \psi_i|^2 + \left(p\alpha - \frac{n(n-4)}{2} \right) |\nabla \psi_i|^2 \right] + \left(p\beta - \frac{n^2(n-4)^2}{16} \right) |\psi|^2 \leq 0.$$

Note the coefficients $p-1$, $p\alpha - \frac{n(n-4)}{2}$ and $p\beta - \frac{n^2(n-4)^2}{16}$ are positive when $\frac{n+4}{n-4} < p < p_c(n)$ where $p_c(n)$ is given by (3.2). \square

Proof of Theorem 3.1. The proof of the Sobolev critical case relies on applying the Pohozaev identity and we omit it here. We now provide a sketch of the proof when $p > \frac{n+4}{n-4}$ in a few steps.

Step 1. $\lim_{r \rightarrow \infty} E(u, r, 0) < \infty$. Note that the energy $E(u, r, 0)$ is nondecreasing in r , as given in Theorem 4.28. Therefore,

$$(3.14) \quad E(u, r, 0) \leq r^{-1} \int_r^{2r} E(u, t, 0) dt \leq r^{-2} \int_r^{2r} \int_t^{t+r} E(u, \lambda, 0) d\lambda dt.$$

Applying estimates given in Corollary 3.1 imply that all of the terms appears in the right-hand side of (3.14) are bounded.

Step 2. Define $u_i^\lambda(x) = \lambda^{\frac{4}{p-1}} u_i(\lambda x)$ for each $1 \leq i \leq m$ where $u = (u_i)_{i=1}^m$ is a stable solution of (1.1). Then $u_i^\lambda \rightarrow u_i^\infty$ where $u_i^\infty \in W_{loc}^{2,2}(\mathbb{R}^n) \cap L_{loc}^{p+1}(\mathbb{R}^n)$ and $u^\infty = (u_i^\infty)_{i=1}^m$ is a stable solution of (1.1).

Set $w_i^\lambda(x) := \lambda^{\frac{4}{p-1}+2} w_i(\lambda x)$. From Corollary 3.1

$$(3.15) \quad \int_{B_r(x)} |w^\lambda|^2 + |u^\lambda|^{p+1} \leq C r^{n-4\frac{p+1}{p-1}}.$$

From elliptic estimates, up to a subsequence, $u_i^\lambda \rightarrow u_i^\infty$ for each $1 \leq i \leq m$ weakly in $W_{loc}^{2,2}(\mathbb{R}^n) \cap L_{loc}^{p+1}(\mathbb{R}^n)$. From compactness embeddings and applying interpolation we arrive at $u_i^\lambda \rightarrow u_i^\infty$ in $L_{loc}^q(\mathbb{R}^n)$ for any $q \in [1, p+1)$. Note also that u^∞ is a stable solution, since u^λ is a stable solution and we can send λ to infinity.

Step 3. u^∞ is a homogeneous solution. This is a direct consequence of the monotonicity formula and the following fact

$$(3.16) \quad \lim_{\lambda \rightarrow \infty} [E(u^\lambda, R, 0) - E(u^\lambda, r, 0)] = 0.$$

The left-hand side of (3.16) is bounded from below by

$$(3.17) \quad E(u^\lambda, R, 0) - E(u^\lambda, r, 0)$$

$$(3.18) \quad \geq C(n, p) \sum_{i=1}^m \int_{B_R \setminus B_r} \left(\frac{4}{p-1} |x|^{-1} u_i^\lambda + \frac{\partial u_i^\lambda}{\partial r} \right)^2 |x|^{\frac{8}{p-1}+2-n} dx$$

$$(3.19) \quad = C(n, p) \sum_{i=1}^m \int_{B_R \setminus B_r} \left(\frac{4}{p-1} |x|^{-1} u_i^\infty + \frac{\partial u_i^\infty}{\partial r} \right)^2 |x|^{\frac{8}{p-1}+2-n} dx.$$

This implies that for each $1 \leq i \leq m$, we have

$$(3.20) \quad u_i^\infty(x) = |x|^{-\frac{4}{p-1}} u_i^\infty\left(\frac{x}{|x|}\right).$$

This finishes the proof of this step.

Step 4. $\lim_{r \rightarrow \infty} E(u, r, 0) = 0$. Note that Theorem 3.2 implies that $u^\infty = 0$, since u^∞ is a homogenous solution. This implies that for each $i = 1, \dots, m$

$$(3.21) \quad \lim_{\lambda \rightarrow \infty} u_i^\lambda = 0,$$

strongly in $L^2(B_4)$. This yields

$$(3.22) \quad \lim_{\lambda \rightarrow \infty} \int_{B_4} |u^\lambda|^2 = 0.$$

This implies that

$$(3.23) \quad \sum_{i=1}^m \lim_{\lambda \rightarrow \infty} \int_{B_4} |u_i^\lambda w_i^\lambda| = 0.$$

From Corollary 3.1 we get This implies that

$$(3.24) \quad \lim_{\lambda \rightarrow \infty} \int_{B_4} |w^\lambda|^2 + |u^\lambda|^{p+1} = 0.$$

On the other hand, there exists r_0 such that

$$\lim_{i \rightarrow \infty} \|u^{\lambda_i}\|_{W^{2,2}(\partial B_{r_0})} = 0.$$

This implies that

$$\lim_{i \rightarrow \infty} E(u, 0, \lambda_i r_0) = \lim_{i \rightarrow \infty} E(u^{\lambda_i}, 0, r_0).$$

Finally, the fact that E is nondecreasing finishes the proof. \square

4. NONLOCAL CASE: LOWER AND HIGHER ORDER FRACTIONAL LAPLACIAN

In this section, we consider system (1.1) with the fractional Laplacian operator $(-\Delta)^s$ where $s \in (0, 2)$ for $s \neq 1$ and we establish Theorem 1.5. We first note that the following Hardy inequality holds for $n > 2s$

$$(4.1) \quad \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{h}|^2 d\xi > \Lambda_{n,s} \int_{\mathbb{R}^n} |x|^{-2s} h^2 dx,$$

for any $h \in C_c^\infty(\mathbb{R}^n)$ where the optimal constant is given by

$$(4.2) \quad \Lambda_{n,s} = 2^{2s} \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

Note that $\Lambda_{n,2} = \frac{n^2(n-4)^2}{16}$ meaning (4.1) recovers (3.7) for $s = 2$. For more information interested readers are encouraged to see [26] by Herbst (and also [43]). We now provide a classification result for homogeneous solutions. Note that for the case of scalar equations, that is when $m = 1$, this classification was given in [21]. Note also that the proof is valid regardless of magnitude of the parameter s .

Theorem 4.1. Suppose $u_i = r^{-\frac{2s}{p-1}}\psi_i(\theta)$ is a stable solution of (1.1) for $s \in (0, 2)$ and $s \neq 1$. Then, each ψ_i vanishes identically, provided $p > \frac{n+2s}{n-2s}$ and

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

Proof. Let $u = (u_i)_{i=1}^m$ be a stable solution of (1.1), and then each ψ_i satisfies

$$\begin{aligned} (4.3) \quad |x|^{-\frac{2ps}{p-1}}|\psi|^{p-1}(\theta)\psi_i(\theta) &= \int \frac{|x|^{-\frac{2s}{p-1}}\psi_i(\theta) - |y|^{-\frac{2s}{p-1}}\psi_i(\sigma)}{|x-y|^{n+2s}} dy \\ &= |x|^{-\frac{2ps}{p-1}} \left[\int \frac{\psi_i(\theta) - t^{-\frac{2s}{p-1}}\psi_i(\theta)}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dt d\sigma \right. \\ &\quad \left. + \int \frac{t^{-\frac{2s}{p-1}}(\psi_i(\theta) - \psi_i(\sigma))}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dt d\sigma \right], \end{aligned}$$

where we have used the change of variable $|y| = t|x|$. Simplifying the above, for each i we obtain

$$(4.4) \quad \psi_i(\theta)A_{n,s} + \int_{\mathbb{S}^{n-1}} K_{\frac{2s}{p-1}}(< \theta, \sigma >) (\psi_i(\theta) - \psi_i(\sigma)) d\sigma = |\psi|^{p-1}(\theta)\psi_i(\theta),$$

where

$$(4.5) \quad A_{n,s} := \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{1 - t^{-\frac{2s}{p-1}}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} d\sigma dt,$$

and

$$(4.6) \quad K_{\frac{2s}{p-1}}(< \theta, \sigma >) := \int_0^\infty \frac{t^{n-1-\frac{2s}{p-1}}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} dt.$$

Note that straightforward calculations show that

$$(4.7) \quad K_{\frac{2s}{p-1}}(< \theta, \sigma >) = \int_0^1 \frac{t^{n-1-\frac{2s}{p-1}} + t^{2s-1+\frac{2s}{p-1}}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} dt,$$

and therefore one can see that $K_\alpha(< \theta, \sigma >) = \int_0^1 \frac{t^{n-1+\alpha} + t^{2s-1+\alpha}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} dt$ is decreasing in α for $p > \frac{n+2s}{n-2s}$. From (4.5) we get the following

$$(4.8) \quad \int_{\mathbb{S}^{n-1}} |\psi|^2(\theta) A_{n,s} + \int_{\mathbb{S}^{n-1}} K_{\frac{2s}{p-1}}(< \theta, \sigma >) |\psi(\theta) - \psi(\sigma)|^2 d\theta d\sigma = \int_{\mathbb{S}^{n-1}} |\psi|^{p+1}(\theta) d\theta.$$

We set a standard cut-off function $\eta_\epsilon \in C_c^1(\mathbb{R}_+)$ at the origin and at infinity that is $\eta_\epsilon = 1$ for $\epsilon < r < \epsilon^{-1}$ and $\eta_\epsilon = 0$ for either $r < \epsilon/2$ or $r > 2/\epsilon$. We now apply the stability inequality (1.24). Set the test function $\phi_i(x) = r^{-\frac{n-2s}{2}}\psi_i(\theta)\eta(r)$ and the fact that $u_i = r^{-\frac{2s}{p-1}}\psi_i(\theta)$ implies that the left-hand side of (1.24) is of the form

$$(4.9) \quad \sum_{i=1}^m \int |u|^{p-1} \phi_i^2 + (p-1) \sum_{i,j=1}^m |u|^{p-3} u_i u_j \phi_i \phi_j = p \int_0^\infty r^{-1} \eta_\epsilon^2(r) dr \int_{\mathbb{S}^{n-1}} |\psi|^{p+1}(\theta) d\theta.$$

We now compute the right-hand side of the stability inequality (1.24) for $\phi_i(x) = r^{-\frac{n-2s}{2}}\psi_i(\theta)\eta_\epsilon(r)$ and $u_i = r^{-\frac{2s}{p-1}}\psi_i(\theta)$. We use the change of variable $|y| = t|x|$ as

following,

$$\begin{aligned}
(4.10) \quad & \int_{\mathbb{R}^n} \frac{\phi_i(x) - \phi_i(y)}{|x - y|^{n+2s}} dy \\
&= r^{-\frac{n}{2}-s} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{\psi_i(\theta)\eta_\epsilon(r) - t^{-\frac{n-2s}{2}}\psi_i(\sigma)\eta_\epsilon(rt)}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dt d\sigma \\
&= r^{-\frac{n}{2}-s} \eta_\epsilon(r) \psi_i(\theta) \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{1 - t^{\frac{n-2s}{2}}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dt d\sigma \\
&\quad + r^{-\frac{n}{2}-s} \eta_\epsilon(r) \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{t^{n-1-\frac{n-2s}{2}}(\psi_i(\theta) - \psi_i(\sigma))}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} dt d\sigma \\
&\quad + r^{-\frac{n}{2}-s} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{t^{n-1-\frac{n-2s}{2}}(\eta_\epsilon(r) - \eta_\epsilon(rt))\psi_i(\sigma)}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} dt d\sigma.
\end{aligned}$$

Set $\Lambda_{n,s} := \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{1-t^{\frac{n-2s}{2}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}} t^{n-1} d\sigma dt$. Then one can conclude that

$$\begin{aligned}
(4.11) \quad & \int_{\mathbb{R}^n} \frac{\phi_i(x) - \phi_i(y)}{|x - y|^{n+2s}} dy \\
&= r^{-\frac{n}{2}-s} \eta_\epsilon(r) \psi_i(\theta) \Lambda_{n,s} \\
&\quad + r^{-\frac{n}{2}-s} \eta_\epsilon(r) \int_{\mathbb{S}^{n-1}} K_{\frac{n-2s}{2}}(< \theta, \sigma >) (\psi_i(\theta) - \psi_i(\sigma)) d\sigma \\
&\quad + r^{-\frac{n}{2}-s} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{t^{-\frac{n-2s}{2}}(\eta_\epsilon(r) - \eta_\epsilon(rt))\psi_i(\sigma)}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} dt d\sigma.
\end{aligned}$$

It is straightforward to show that the following holds for the right-hand side of (1.24),

$$\begin{aligned}
(4.12) \quad & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\phi_i(x) - \phi_i(y))^2}{|x - y|^{n+2s}} dx dy \\
&= 2 \int_0^\infty r^{-1} \eta_\epsilon^2(r) dr \int_{\mathbb{S}^{n-1}} \psi_i^2 \Lambda_{n,s} d\theta \\
&\quad + 2 \int_0^\infty r^{-1} \eta_\epsilon^2(r) dr \int_{\mathbb{S}^{n-1}} K_{\frac{n-2s}{2}}(< \theta, \sigma >) (\psi_i(\theta) - \psi_i(\sigma))^2 d\sigma d\theta \\
&\quad + 2 \int_0^\infty \left[\int_0^\infty r^{-1} \eta_\epsilon(r) (\eta_\epsilon(r) - \eta_\epsilon(rt)) dr \right] \\
&\quad \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{t^{n-1-\frac{n-2s}{2}} \psi_i(\sigma) \psi_i(\theta)}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} d\sigma d\theta dt.
\end{aligned}$$

Applying similar idea provided in [21], we collect higher order terms of the stability inequality to get

$$(4.13) \quad \Lambda_{n,s} \int_{\mathbb{S}^{n-1}} |\psi|^2 + \int_{\mathbb{S}^{n-1}} K_{\frac{n-2s}{2}}(< \theta, \sigma >) |\psi(\theta) - \psi(\sigma)|^2 d\sigma \geq p \int_{\mathbb{S}^{n-1}} |\psi|^{p+1}.$$

Combining this and (4.8) we end up with

$$(4.14) \quad (\Lambda_{n,s} - pA_{n,s}) \int_{\mathbb{S}^{n-1}} |\psi|^2 + \int_{\mathbb{S}^{n-1}} (K_{\frac{n-2s}{2}} - pK_{\frac{2s}{p-1}})(< \theta, \sigma >) |\psi(\theta) - \psi(\sigma)|^2 d\sigma \geq 0.$$

The fact that K_α is decreasing in α implies $K_{\frac{n-2s}{2}} < K_{\frac{2s}{p-1}}$ for $p > \frac{n+2s}{n-2s}$. Therefore, $K_{\frac{n-2s}{2}} - pK_{\frac{2s}{p-1}} < 0$. On the other hand the assumption of the theorem implies that $\Lambda_{n,s} - pA_{n,s} < 0$. Therefore, each ψ_i vanishes identically. This finishes the proof. \square

4.1. Lower Order Fractional Laplacian Operator. In this part, we show that Theorem 1.5 holds when $0 < s < 1$. To do so we provide the following estimate first.

Lemma 4.1. *Suppose that $p \neq \frac{n+2s}{n-2s}$. Let u be a solution of (1.1) that is stable outside a ball B_{R_0} and v satisfies (1.11). Then there exists a constant $C > 0$ such that*

$$(4.15) \quad \int_{B_R} y^{1-2s} |v|^2 \leq CR^{n+2-2s\frac{p+1}{p-1}},$$

for any $R > 3R_0$.

Lemma 4.2. *Let u be a solution of (1.1) that is stable outside a ball B_{R_0} and v satisfies (1.11). Then there exists a positive constant C such that*

$$(4.16) \quad \int_{B_R \cap \partial \mathbb{R}_+^{n+1}} |v|^{p+1} + \sum_{i=1}^m \int_{B_R \cap \mathbb{R}_+^{n+1}} y^{1-2s} |\nabla v_i|^2 \leq CR^{n-2s\frac{p+1}{p-1}}.$$

Proof of Theorem 1.5 when $0 < s < 1$. We omit the proof here since arguments are very similar to the ones which will be provided for the case of $1 < s < 2$. \square

4.2. Higher Order Fractional Laplacian Operator. As the last part of this section, we shall restrict ourselves to the case $1 < s < 2$. Let us start with the following integral estimate on stable solutions.

Lemma 4.3. *Let u be a solution of (1.1) that is stable outside a ball B_{R_0} and v satisfies (1.13). Then there exists a positive constant C such that*

$$(4.17) \quad \int_{\partial \mathbb{R}_+^{n+1}} |v|^{p+1} \eta^2 + \sum_{i=1}^m \int_{\mathbb{R}_+^{n+1}} y^b |\Delta_b v_i|^2 \eta^2$$

$$(4.18) \quad \leq C \int_{\mathbb{R}_+^{n+1}} y^b |v|^2 (|\Delta_b \eta|^2 + |\Delta_b |\nabla \eta|^2| + |\nabla \eta \cdot \nabla \Delta_b \eta|)$$

$$(4.19) \quad + C \sum_{i=1}^m \int_{\mathbb{R}_+^{n+1}} y^b |v_i| |\Delta_b v_i| |\nabla \eta|^2.$$

Proof. Here is a sketch of the proof. Multiply the i^{th} equation of (1.13) with $y^b u_i \eta^2$ where η is a test function to get

$$(4.20) \quad C_{n,s} \int_{\partial \mathbb{R}_+^{n+1}} |v|^{p+1} \eta^2 = \sum_{i=1}^m \int_{\mathbb{R}_+^{n+1}} y^b \Delta_b v_i \Delta_b (v_i \eta^2).$$

Note also that for each i these identities hold for a test function η

$$\begin{aligned} \Delta_b v_i \Delta_b (v_i \eta^2) - |\Delta_b (v_i \eta)|^2 &= -v_i^2 |\Delta_b \eta|^2 + 2v_i \Delta_b v_i |\nabla \eta|^2 - 4|\nabla v_i \cdot \nabla \eta|^2 \\ &\quad - 4v_i \Delta_b \eta \nabla v_i \cdot \nabla \eta, \\ \Delta_b (v_i \eta) &= \eta \Delta_b v_i + v_i \Delta_b \eta + 2\nabla v_i \cdot \nabla \eta. \end{aligned}$$

Applying these identifies together with (4.20) one can see that

$$\begin{aligned}
 C_{n,s} \int_{\partial \mathbb{R}_+^{n+1}} |v|^{p+1} \eta^2 &= \sum_{i=1}^m \int_{\mathbb{R}_+^{n+1}} y^b |\Delta_b(v_i \eta)|^2 + 2 \sum_{i=1}^m \int_{\mathbb{R}_+^{n+1}} y^b v_i \Delta_b v_i |\nabla \eta|^2 \\
 (4.21) \quad &- 4 \sum_{i=1}^m \int_{\mathbb{R}_+^{n+1}} y^b |\nabla v_i \cdot \nabla \eta|^2 + \int_{\mathbb{R}_+^{n+1}} y^b |v|^2 (|\Delta_b \eta|^2 + 2 \nabla \eta \cdot \nabla \Delta_b \eta).
 \end{aligned}$$

Testing the stability inequality (1.24) on $\phi_i = u_i \eta$ and applying (4.21) one can finish the proof. \square

One can set η to be the standard test function to get the following estimate.

Corollary 4.1. *With the same assumption as Lemma 4.3. Then there exists a positive constant C such that*

$$(4.22) \quad \int_{B_R \cap \partial \mathbb{R}_+^{n+1}} |v|^{p+1} + \sum_{i=1}^m \int_{B_R \cap \mathbb{R}_+^{n+1}} y^b |\Delta_b v_i|^2 \leq CR^{-4} \int_{B_R \cap \mathbb{R}_+^{n+1}} y^b |v|^2.$$

Here we provide more decay estimates of solutions. These lemmata are main tools in our proof of Theorem 1.5. We omit the proofs here and we refer interested readers to see the proof of Lemma 4.5-4.6 in [21] where similar arguments are applied.

Lemma 4.4. *Suppose that $p \neq \frac{n+2s}{n-2s}$. Let u be a solution of (1.1) that is stable outside a ball B_{R_0} and v satisfies (1.13). Then there exists a constant $C > 0$ such that*

$$(4.23) \quad \int_{B_R} y^b |v|^2 \leq CR^{n+4-2s\frac{p+1}{p-1}},$$

for any $R > 3R_0$.

Lemma 4.5. *Let u be a solution of (1.1) that is stable outside a ball B_{R_0} and v satisfies (1.13). Then there exists a positive constant C such that*

$$(4.24) \quad \int_{B_R \cap \partial \mathbb{R}_+^{n+1}} |v|^{p+1} + \sum_{i=1}^m \int_{B_R \cap \mathbb{R}_+^{n+1}} y^b |\Delta_b v_i|^2 \leq CR^{n-2s\frac{p+1}{p-1}}.$$

Proof of Theorem 1.5 when $1 < s < 2$. The proof of the case $p = p_S(n, s)$ is based on the Pohozaev identity provided in [21, 35], and we omit it here. The proof is based on the monotonicity formula that is Theorem 1.4 and a blow-down analysis.

Step 1. The energy is bounded, that is $\lim_{\lambda \rightarrow \infty} E(v, 0, \lambda) < \infty$. This is a direct consequence of the monotonicity formula and Lemma 4.4 and Lemma 4.5. Similar to the proof of Theorem 3.1 we have

$$(4.25) \quad E(v, \lambda, 0) \leq \lambda^{-2} \int_{\lambda}^{2\lambda} \int_t^{t+\lambda} E(v, \gamma, 0) d\gamma dt.$$

Lemma 4.5 and Lemma 4.4 imply that the right-hand side of (4.25) is bounded.

Step 2. The sequence v_i^λ converges weakly in $H_{loc}^1(\mathbb{R}^n, y^{3-2s} dx dy)$ to a function v_i^∞ where each v_i^∞ is homogeneous for $1 \leq i \leq m$ and therefore they are zero.

Note that the convergence part is a direct consequence of the elliptic estimates. We now show that each v_i^∞ is homogeneous. From the boundedness of the energy we have

$$(4.26) \quad \lim_{k \rightarrow \infty} [E(v, R_2 \lambda_k, 0) - E(v, R_1 \lambda_k, 0)] = 0.$$

From this and applying the scaling invariance of the energy and also the monotonicity formula we get

$$\begin{aligned} 0 &= \liminf_{k \rightarrow \infty} \sum_{i=1}^m \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}_+^{n+1}} y^{3-2s} r^{\frac{4s}{p-1} + 2s-2-n} \left(\frac{2s}{p-1} r^{-1} v_i^{\lambda_k} + \frac{\partial v_i^{\lambda_k}}{\partial r} \right)^2 dy dx \\ &\geq \sum_{i=1}^m \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}_+^{n+1}} y^{3-2s} r^{\frac{4s}{p-1} + 2s-2-n} \left(\frac{2s}{p-1} r^{-1} v_i^\infty + \frac{\partial v_i^\infty}{\partial r} \right)^2 dy dx, \end{aligned}$$

since we have the weak convergence of $(v_i^{\lambda_k})$ to v_i^∞ in $H_{loc}^1(\mathbb{R}^n, y^{3-2s} dy dx)$. Therefore,

$$\frac{2s}{p-1} r^{-1} v_i^\infty + \frac{\partial v_i^\infty}{\partial r} = 0 \quad \text{a.e. in } \mathbb{R}_+^{n+1},$$

for each $1 \leq i \leq m$.

Step 3. $\lim_{\lambda \rightarrow \infty} E(v, \lambda, 0) = 0$. Note that the monotonicity formula implies that

$$(4.27) \quad E(v, \lambda, 0) \leq \lambda^{-1} \int_\lambda^{2\lambda} E(t) dt \leq \sup_{[\lambda, 2\lambda]} I + C \lambda^{-n-1 + \frac{2s(p+1)}{p-1}} \int_{B_{2\lambda} \setminus B_\lambda} |v|^2,$$

where

$$\begin{aligned} I(v, \lambda) &= I(v^\lambda, 1) \\ &= \frac{1}{2} \sum_{i=1}^m \int_{\mathbb{R}_+^{n+1} \cap B_1} y^{3-2s} |\Delta_b v_i^\lambda|^2 dx dy \\ &\quad - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |v^\lambda|^{p+1} dx. \end{aligned}$$

Note that $\lim_{\lambda \rightarrow \infty} I(v, \lambda) = 0$. On the other hand, from the fact that u^∞ is a homogenous solution we have

$$(4.28) \quad \lim_{\lambda \rightarrow \infty} u_i^\lambda = 0,$$

strongly in $L^2(B_4)$. Therefore,

$$(4.29) \quad \lim_{\lambda \rightarrow \infty} \int_{B_4} |u^\lambda|^2 = 0.$$

This implies that $\lim_{\lambda \rightarrow \infty} E(v, \lambda, 0) = 0$ and this finishes the proof. \square

At the end, let us mention that in an upcoming article [20] we apply monotonicity formulae derived in here to prove regularity of free boundaries and partial regularity of weak solutions for certain coupled elliptic systems.

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